A Study of Bose-Einstein Condensation in a Two-Dimensional Trapped Gas

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Abstract

We examine the possibility of Bose-Einstein condensation (BEC) in two-dimensional (2D) system of interacting particles in a trap. We use a self-consistent mean-field theory of Bose particles interacting by a contact interaction in the Popov and WKB approximations. The equations show that the normal state has a phase transition at some critical temperature T_c but below T_c the Bose-Einstein condensed state is *not* a consistent solution of the equations in the thermodynamic limit. This result agrees with a theorem recently discussed by the author that shows that a BEC state is impossible for an interacting gas in a 2D trap in the thermodynamic limit.

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I. Introduction

Recent experiments on alkali atoms[1] confined in three-dimensional (3D) magnetic traps and cooled by evaporation techniques have led to the observation of Bose-Einstein condensation (BEC). In these experiments the number of particles cooled has ranged from several thousand to millions. Experiments to date have naturally been in 3D systems. If experimentally it becomes possible to make one dimension of the trap so narrow that the harmonic states are very greatly separated, then the system could be a reasonable simulation of a two-dimensional system. Further, there is the possibility of an adsorbed gas, such as spin-polarized hydrogen on liquid helium,[2] forming a 2D sys-

tem.

The ideal Bose gas in two dimensions trapped in a harmonic potential *has* a Bose-Einstein condensation (BEC).[3, 4] However, the author[4] recently demonstrated how theorems by Hohenberg and Chester could be used to show that the trapped system can have no BEC, in the thermodynamic limit, if there are interactions that prevent the density from diverging at any position. The ideal gas has a divergence at the origin at and below the transition temperature, which excludes it from application of the theorem, and a BEC does occur.

Our intent here is to test the general theorem by an explicit mean-field computation of the gas properties when interactions are present. We consider particles interacting by a contact potential. In 3D such an interaction is a pseudo-potential for a hard-core interaction. This is not the case in 2D, however it provides a simple model that can be analyzed by mean-field theory.

Recently Griffin[5] discussed the generalized finite-temperature Hartree-Fock-Bogoliubov (HFB) equations of motion for a Bose gas in a nonuniform potential and the various approximations for treating it. He points out that a particularly useful gapless approximation is that of Popov. Giorgini *et al* (GPS)[6] have solved the equations in this form for three dimensions (3D) by use of the WKB approximation. Hutchinson *et al* [7] have solved the equations directly.

We study the 2D system in the thermodynamic limit where the condensate equation may be treated exactly by the Fermi-Thomas approach. As in the ideal gas case, in the equations without condensate, there is a critical temperature below which no value of the chemical potential satisfies the condition on the number of particles. At that point one usually invokes the presence of a condensate to satisfy the particle-number condition. However,

we find that *there is no solution to the self-consistent equations for condensate and excited-state particles below the critical temperature.* Apparently the noncondensed system becomes unstable at the critical temperature and makes a transition to some other state, but the state is *not* the BEC state. Possibly there is a Kosterlitz-Thouless transition,[2] but we have not yet checked that hypothesis.[8]

II. THE THERMODYNAMIC LIMIT FOR HARMONIC TRAPS

Consider a 2D system of N Bose particles in a spherically symmetric harmonic trap. The harmonic potential is given by

$$U(\mathbf{r}) = \frac{1}{2}U_0(\frac{r}{R})^2 \tag{1}$$

where $r^2 = x^2 + y^2$ and R is a range parameter that will be handy for taking the thermodynamic limit. The angular frequency is

$$\omega = \sqrt{\frac{U_0}{R^2 m}} \tag{2}$$

where m is the particle mass.

If we wish to take the thermodynamic limit we must increase the "volume" while keeping the average density fixed. The average density is proportional to $\rho = N/R^2$ where R is the range parameter in Eq. (1) chosen at some convenient temperature to be a distance within which the majority of particles resides. Increasing the volume then implies weakening the potential, by increasing R, while N increases. From Eq. (2) we see that this requires keeping $N\omega^2 = const$ while $N \to \infty$, This limiting process has been considered previously.[3, 4, 9] Define a characteristic temperature

$$T_0 = \frac{\hbar}{k_B}\sqrt{\frac{U_0\rho}{m}} \tag{3}$$

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where k_B is the Boltzmann constant. We see that $k_B T_0 = \sqrt{N}\hbar\omega$ remains constant as the thermodynamic limit is taken. For the ideal gas in a harmonic trap there is a phase transition[3, 4] at $T_c = T_0/\sqrt{\zeta(2)}$ where $\zeta(\sigma)$ is the Riemann ζ -function.

III. HARTREE-FOCK-BOGOLIUBOV EQUATIONS

Recently Griffin[5] has discussed the derivation of a self-consistent mean-field treatment of the inhomogeneous interacting Bose gas valid at finite temperatures. One writes the field operator for the bosons as

$$\hat{\psi} = \Phi + \tilde{\psi} \tag{4}$$

where $\Phi = \langle \hat{\psi} \rangle$ is the condensate wave function and $\tilde{\psi}$ describes fluctuations. What results in the Popov approximation is a generalized Gross-Pitevskii equation valid at $T > 0$ that now depends not only on the local condensate density $n_0(\mathbf{r}) = \langle \Phi^* \Phi \rangle$ but also on the density $n_T(\mathbf{r}) = \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle$ of particles in excited states. When we consider a contact interaction of strength g, the equation for the condensate is[5]

$$[\Lambda - gn_0(\mathbf{r})]\,\Phi = 0 \tag{5}$$

where the operator Λ is

$$\Lambda = -\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) - \mu + 2gn(\mathbf{r}), \tag{6}$$

μ is the chemical potential, and $n = n_0 + n_T$ is the total local density.

There is also an equation for $\tilde{\psi}$ that depends on n_0 and n_T. A Bogoliubov transformation of this equation leads to a pair of differential equations, which Hutchinson *et al* [7] have solved in 3D by introducing an eigenfunction basis.

Giorgini *et al* [6] have used the WKB approximation to simplify and solve the equations for the excitations in 3D. The result of the same procedure in 2D is the following: The excitation spectrum is

$$\epsilon(\mathbf{p}, \mathbf{r}) = \sqrt{\bar{\Lambda}^2 - (gn_0)^2} \tag{7}$$

where $\bar{\Lambda} = \frac{p^2}{2m} + U(\mathbf{r}) - \mu + 2gn(\mathbf{r})$. The density of the excited particles is given by

$$n_T(\mathbf{r}) = \frac{1}{h^2} \int d\mathbf{p} \left\{ \left[u^2(\mathbf{p}, \mathbf{r}) + v^2(\mathbf{p}, \mathbf{r}) \right] f(\mathbf{p}, \mathbf{r}) + v^2(\mathbf{p}, \mathbf{r}) \right\} \tag{8}$$

with

$$f(\mathbf{p}, \mathbf{r}) = \frac{1}{e^{\beta \epsilon} - 1} \tag{9}$$

and

$$u^2(\mathbf{p}, \mathbf{r}) = \frac{\bar{\Lambda} + \epsilon}{2\epsilon} \tag{10}$$

$$v^2(\mathbf{p}, \mathbf{r}) = \frac{\bar{\Lambda} - \epsilon}{2\epsilon} \tag{11}$$

The total number of particles satisfies

$$N = \int d\mathbf{r} \left[n_0(\mathbf{r}) + n_T(\mathbf{r}) \right] \tag{12}$$

The semiclassical approach makes sense if $k_B T \gg \hbar \omega$.

IV. A TRANSITION, BUT TO WHAT STATE?

In 2D Eq. (8) can be integrated. When one changes variables from p to $y = \beta \epsilon$ the integral takes on a particularly simple form:

$$\begin{aligned}
n_T(\mathbf{r}) &= \frac{1}{\lambda^2} \int_{\sqrt{t^2 - s^2}}^{\infty} dy \left[\frac{1}{e^y - 1} + \frac{1}{2} \left(1 - \frac{y}{\sqrt{y^2 + s^2}} \right) \right] \\
&= \frac{1}{\lambda^2} \left\{ -\ln \left[1 - \exp(-\sqrt{t^2 - s^2}) \right] + t - \sqrt{t^2 - s^2} \right\} \tag{13}
\end{aligned}$$

where $t = \beta\left(U(\mathbf{r}) - \mu + 2gn(\mathbf{r})\right)$, $s = \beta g n_0(\mathbf{r})$, and $\lambda^2 = h^2/2\pi m k_B T$.

When there is no condensate the density is simply

$$n_T^{(>)}(\mathbf{r}) = -\frac{1}{\lambda^2}\ln\left[1 - \exp(-t)\right] \tag{14}$$

We have solved Eq. (14) self-consistently for μ by numerical means and find that, just as in the ideal gas case, there is a solution only for T greater that some critical value. Sample results are shown in Fig. 1. Below the critical temperature we expect a new phase to exist. To see if the transition is to the BEC state we must consider the full set of equations.

Baym and Pethick[10] have shown that the Fermi-Thomas approximation in which the kinetic energy is neglected is valid when N is large. In our 2D case one can show that the kinetic energy diminishes relatively as $1/N$. Thus in the thermodynamic limit the Fermi-Thomas approximation is exact and Eq. (5) leads to

$$n_0 = \frac{1}{g}\left[\mu - U - 2gn_T\right] \tag{15}$$

This equation along with Eqs. (12) and (13) must be solved self-consistently. However, we see immediately that this is impossible because from Eq. (15), $s = t$, and the exponential in Eq. (13) vanishes giving nonsense for all conditions. What has happened is that the lower limit in the integral form of Eq. (13) has vanished indicating that the long-wavelength phonons have destabilized long-range order just as in the homogeneous case. The BEC equations are inconsistent in agreement with the theorem discussed in Ref. 4, and there is no BEC in 2D in the thermodynamic limit. Note however that the mean-field equations for high temperature predict that there *is* a phase transition at some critical temperature. If those equations have any validity in describing a real system they tell us that the normal state of the interacting gas becomes unstable at some temperature. However, what state becomes

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stable is not apparent from the present discussion, possibly a Kosterlitz-Thouless transition occurs.[8]

Real experiments are not done in the thermodynamic limit but with a finite number of particles. There can be a pseudo-condensation, a macroscopic number of particles in the lowest state below some temperature that would go to zero in the thermodynamic limit.[4, 11] Experiments in which this "transition temperature" is tracked as a function of N might be possible. Further theoretical computations for finite N are in order.

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Figure Caption

Fig. 1. $\alpha = -\mu/k_B T$ versus temperature divided by T_0 for a sample set of parameters. Solid line: interacting gas with parameters $\gamma(\equiv g\rho^2/k_B T_0) = 1$ and $\tau_0(\equiv k_B T_0 \rho^2 m/\hbar^2) = 1$. Dotted line: Non-interacting case ($\gamma = 0, \tau_0 = 1$). With interactions there is a solution for μ above a critical reduced temperature, τ_c, but below, the self-consistent equations have no solution indicating there is no long-range condensate order.